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Edited by

JOHN R. CLARK, Editor-in-Chief
EUGENE R. SMITH, Associate Editor

ALFRED DAVIS
MARIE GUGLE
HARRY D. GAYLORD
JOHN W. YOUNG
JOHN A. FOBERG

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JOHN R. CLARK, Editor-in-Chief
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THE MATHEMATICS TEACHER

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YOU KNOW WHAT I MEAN[1]

By RAYMOND K. MORLEY
Worcester Polytechnic Institute

The proper way to begin a story is "Once upon a time," so I shall start mine that way.

Once upon a time there was a mathematician. He probably would not be rated as a great mathematician, although a keen one. In his spare time he made some fairy tales to amuse some children and these were so good that they were published, and brought him much more fame, and I dare say income, than he ever gained as a mathematician. His name was Charles Ludwig Dodgson but you will recognize him much better by the pen-name he used for his fairy tales—Lewis Carroll—whose derivation from the other you will perceive. The books are the Alice books, "Alice in Wonderland" and "Through the Looking-glass." You read them with delight as fairy stories, I suppose, when you were a child, and probably you re-read them afterwards for their wit.

I want to quote a little from "Alice in Wonderland." The scene is the Mad Tea-Party; present, the Mad Hatter, the Mad March Hare, the persistently dozing Dormouse, and Alice.

"I'm glad they've begun asking riddles—I believe I can guess that," she added aloud.

"Do you mean that you think you can find out the answer to it?" said the March Hare.

"Exactly so," said Alice.

"Then you should say what you mean," said the March Hare

"I do," Alice hastily replied; "at least—at least I mean what I say—that's the same thing, you know."

"Not the same thing a bit!" said the Hatter. "Why, you might just as well say that 'I see what I eat' is the same thing as 'I eat what I see'!"

"You might just as well say," added the March Hare, "that 'I like what I get' is the same thing as 'I get what I like'!"

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[1]Read before The Association of Teachers of Mathematics in New England, Cambridge, Mass., May 1, 1926.

"You might just as well say," added the Dormouse, which seemed to be talking in its sleep, "that 'I breathe when I sleep' is the same thing as 'I sleep when I breathe'!"

"It is the same thing with you," said the Hatter, and here the conversation dropped.

Their argument is about the difference between saying what we mean and meaning what we say. Think of arguing over that! But then, of course, it was a Mad Tea-party! Most of us are happy if we succeed in doing either thing, either saying what we mean or meaning what we say. And ordinary language frequently does neither. You know countless examples. Pick up almost any reading matter and you'll find something like "His blood literally boiled"—yet he survived. A few years ago a common advertisement was:

"All soap powders are not Soapine," which, if it means what it says tells us that soap powders and Soapine are two different things—but they are not. It would have been as easy to say "Not all soap powders are Soapine," which would have been exact, but why bother to be exact?

Or Shakespeare—"All that glitters is not gold" which, of course, isn't what he means. But then Shakespeare has a poet's license to do such things. The rest of us haven't that excuse. We seem to have a sort of futurist or cubist attitude towards our language, it doesn't resemble what it's meant for, particularly, but there are certain lines or expressions that somehow convey a general impression, and as to the rest of it—"oh well, you know what I mean!"

Do you think that I exaggerate? It is perfectly easy to test it. Try getting a class, some day, to define something for you, something they are familiar with, but for which they have had no book definition. It doesn't have to be mathematical. Will you get a complete, holeproof definition the first time? Possibly, but eight to five that you do not, would be comfortably safe odds.

Suppose you tried, say, "Hat." Wouldn't you most likely get "Oh, you know, something on your head." And if you venture to suggest that this might equally well be hair or ears, you'll hear with a giggle, "You know what I mean."

And it isn't just our students. We all do it. No one has any patience with saying what we mean or meaning what we say.

If you try to insist on it, you are a pedant, or a lawyer, or something else equally bad. But perhaps I should not talk of such matters here. We are not an association of teachers of English, but of mathematics. And when it comes to exact expression the mathematician, like the Pharisee, flatters himself that he is not as other men are. He has his failings, perhaps, when he talks about coal, or the late winter, or spring fashions, or golf, but as a mathematician the exactness of his expression is above reproach. Let us see.

Consider text-books for example. Have you not often found places where the author depends on "You know what I mean?" It seems to be in the problems especially that one encounters it. Haven't you every now and then worked a problem under two or more interpretations to discover if possible from his answer which the author intended? I heard recently of an instance of poetic retribution in a case of this kind. An author was visiting a class that was using his text-book when they found a problem whose meaning was obscure—to them at least. The class, and probably the visitor, were greatly astonished when the teacher turned to him and asked him to explain what he really did mean by it!

But I want more particularly to call your attention to difficulties of a little different sort, difficulties of meaning in our mathematical system. Let me offer some examples in detail.

Quite early in Algebra we begin to insist that juxtaposition means multiplication, ax means a times x, $3x$ means 3 times x, $3\frac{x}{y}$ means 3 times $\frac{x}{y}$, $3\frac{1}{y}$ means 3 times $\frac{1}{y}$, $3\frac{1}{2}$ means 3 times—but no, no! This one doesn't mean 3 times at all, it means 3 *and* a half! But, you reply, the inconsistency doesn't do any harm, the context shows what is meant—in short, "You know what I mean." Nevertheless, it does sometimes lead students astray. Are they to blame? Of course they are. It is perfectly well known that a reason is no excuse.

Let us look further.

The word "limit" in ordinary language means a boundary, a terminus, an extreme value. But in mathematics, first perhaps in plane geometry, later with greater care in calculus, we give it

a precise, technical meaning, peculiar to the mathematician's trade. It is no longer a boundary necessarily, but something different. It may be defined thus:

When a variable, x, approaches in value a constant, a, in such a way that the numerical value of their difference, $x - a$, eventually becomes less and remains less than any magnitude that can be named, however small, a is defined to be the limit of x, and x is said to approach a as a limit.

You see according to this the variable can, perhaps, hop all around its limit, which may therefore not be a boundary at all. Thus, if the n^{th} value of a variable, v is $v = \dfrac{\sin n\pi/4}{n}$ then as n increases indefinitely limit $v = o$, but v takes on values both sides of the limit, and on the limit. It isn't at all a limit in the ordinary language sense.

There isn't any reason against mathematics' defining the word in its own peculiar fashion, if it likes. Many other trades take common words and give them a meaning of their own.

For instance, take the word "dog," certainly a sufficiently well known word. Generally a dog is an animal, chiefly famous for making trouble between neighbors. But try it in the machine shop; there a dog is entirely different, a sort of clamp, or catch. Or talk about a dog at the race track and you will find it something still different—some kind of a horse, I believe, I think a kind not particularly admired. So the mathematician is perfectly entitled to let limit have its special mathematical meaning.

But then, after the learner has heard all this about limits, and has found that limit for mathematical purposes has a very precise technical definition, he gets along further, meets definite integrals, and finds, also in the mathematical trade, that limit is suddenly restored to its old meaning boundary! Thus he reads $\int_a^b f(x)dx$, "the integral between the limits a and b of $f(x)dx$," and here limit means bound. And to be sure that he is well muddled, he has fired at him both meanings in the same proposition, thus:

$$\text{Limit} \sum f(x)dx = \int_a^b f(x)dx$$

Isn't that the limit?

Here's another example: Among mathematical operations we talk about "cancelling." It is not perhaps so well recognized an operation as addition or multiplication, but it is a term that is used—and used almost as recklessly by the mathematician as it was by the business man during the post-war depression. Sometimes it means "dividing by a common factor" and sometimes "taking away terms whose sum is zero." Furthermore, here we begin to get into ambiguous symbolism as well as language, for both these operations are commonly shown alike with the stroke.

Thus $$\frac{\cancel{(x-1)}\ (x+2)}{(x-3)\ \cancel{(x-1)}}$$

or $$x^2 + \cancel{2x} + 3 - \cancel{2x}$$

We consider a student's cancelling terms in a fraction the high tide of stupidity. But why shouldn't he, with this notation? As I said before, a reason is no excuse. He should know what we mean.

It is perfectly easy to use different notations for those two different operations. Why not, say, reserve the stroke / for dividing out a factor, since it resembles the solidus (as in 1/3) which means division, and use something else for subtraction of equals? A writer in the Mathematics Teacher sometime ago suggested the cross × for this. Personally, I prefer the circle thus: $x^2 - 2x + 2x$, as I use the cross through work to denote wrong. I think many other teachers use it so, also.

As to language for it, probably the safest plan would be to abandon "cancel" entirely and say, "divide out" and "subtract off." The inertia to overcome to do this is considerable, particularly as cancel is used with the former meaning in arithmetic. Might we not then attempt to restrict the word, cancel, like the stroke, to division, and adopt some other, as "subtract off" for removing a zero sum?

But perhaps this matter of "cancel" is hardly fair. Cancelling is a rather informal operation, used by everybody, but perhaps without official sanction. So next I'll take something very official indeed.

I quote from "The Reorganization of Mathematics in Secondary Education—A Report by the National Committee on Mathematical Requirements." In the Chapter on Terms and Symbols in Elementary Mathematics, occurs the following:

"C. Definite usage recommended—

"1. *Circle* should be considered as the curve; but where no ambiguity arises, the word "circle" may be used to refer either to the curve or to the part of the plane inclosed by it."

Isn't that, "Oh well, you know what I mean!" and confessing it?

It goes on likewise:

"2. Polygon (including triangle, square, parallelogram, and the like) should be considered, by analogy to a circle, as a closed broken line; but where no ambiguity arises, the word polygon may be used to refer to the broken line, or to the part of the plane inclosed by it. . . .

"4. *Solids.* The usage above recommended with respect to plane figures is also recommended with respect to solids. For example, *Sphere* should be as a surface, . . . etc., . . . the double use of the word should be allowed where no ambiguity arises."

It is not my intention to dispute the decision of the Committee on these definitions—though I confess that square as a line, rendering meaningless the universal use of the word as an exponent, as x^2, goes hard against the grain. Their verdict is perhaps the best summary possible of divergent usage. It is the usage that is at fault.

It is a well known principle that when the doctors disagree the patient dies. For us the student is the patient. It is all very well to say, "where no ambiguity arises,"—you know what I mean. But does the student know what we mean, when we have defined something one way and then use it another? You may have warned him of the new meaning, but the old one will stick and bob up to confuse him. How many times have I labored with college freshmen over cylinders in Analytic Geometry! We have redefined them, carefully, no bases, directed by a curve not even necessarily closed. Then we meet them again a little later,

and some one is sure to turn up with his cylinder still the right circular cylinder of solid geometry.

The point is that there is a real reason for such confusion, a reason that lies in our mathematical system. Of course the context shows which is the correct interpretation in any given case, but to guess the answer to "you know what I mean" requires an added strain on the student's mental agility when he needs it all for other things.

But perhaps this isn't exactly fair to mathematics. Mathematics acquires its reputation for exact expression less, perhaps, from the words it uses than from its symbols. It is there that we should look for the precise expression of mathematics. Indeed, so strongly has the need of dispensing with words, possibly ambiguous, in mathematics been felt that systems have been devised that do away with words altogether. Probably the best known is the system of Peano. Whether the use of Peanian would remove all our difficulties of meanings, is doubtful.

Henri Poincaré, in his lifetime one of the greatest mathematicians of his day, also a great philosopher, and a great physicist, rather pokes fun at these symbolists. He says, of a treatise written in this language, "I ask definitely whether this form makes it gain much in the way of exactness, and whether it thereby compensates for the efforts it imposes upon the writer and the reader." He then proceeds to catch the author he is discussing in a petitio principii. So symbols alone do not cure inaccuracy.

But I think I can make my point without Peanian. Perhaps our commonest mathematical symbol is the equals mark $=$. It seems simple and definite. But it has two meanings at least. Thus $x^2 + x = 6$ and $(x - a)\ (x + a) = x^2 - a^2$. In the first it means "shall equal" or " is to equal," while in the second it means "equals" or "equals identically." It is true that these meanings are not widely different, but they do at times cause confusion.

There is, of course, another symbol for the second meaning $\equiv$ which is defined as meaning "equals identically," but it is very seldom used. We generally prefer to fall back on "Oh well, you know what I mean!" One trigonometry confesses it frankly

thus: "The symbol denoting identity is $\equiv$. When there can be no misunderstanding as to the meaning, the sign of equality is used to denote identity."

It is certainly true that these two meanings of the equality symbol often shade into one another so that it might be difficult sometimes to decide which we have.[1] But any one who has taught trigonometry has found students confused by these two meanings.

Furthermore, there are other difficulties with this symbol for identity. In the theory of numbers it is used for something quite different, the congruence. For example, $28 \equiv 1 \pmod 3$ is read 28 is congruent to 1, modulus 3, and means that if you divide 28 by 3 the remainder is 1. The statement of the modulus is sometimes left out, leaving the symbol identical for the two meanings.

Doesn't it seem as if, on occasions when a possible doubt as to meaning requires a symbol to supplement the equality sign and make it more precise, something might be found for the purpose that does not run into added confusion on this other score?

Then, too, there is another language difficulty in this word congruent. In plane geometry you know it has a meaning quite different from that in the theory of numbers just mentioned. It means superposable, that is, if you allow that in our two-dimensional geometry we can turn figures over through a third dimension. We don't allow the corresponding thing in solid figures. Reversed solids are only symmetrical, not congruent. So in two branches of mathematics we have two different, technical meanings of this word congruent. Of course, language is full of examples of words with many meanings, but isn't it a bit unfortunate that when the term "equal," for such figures, which I was brought up on, gave way in the interest of exactness to a more specialized term it could not have been one not already in use elsewhere?

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[1] For instance, $\sin^2 \Theta + \cos^2 \Theta = 1$, is an identity being true for all values of Θ. But suppose $\cos \Theta = 3/5$ is given; then this becomes $\sin^2 \Theta + (3/5)^2 = 1$, an equation to be solved for $\sin \Theta$ and true for only two values of it.

I do not think this list of symbols with confusing meanings is exhaustive but perhaps it is long enough to illustrate my point.[1]

Of course the difficulties come largely from the growth of mathematics. Former meanings, later displaced or modified, gave rise to symbols or words that no longer fit exactly.

For instance, take the word limit, which I mentioned before. Probably it did originally mean a boundary. The precise, highly technical meaning in the definition is a much later development. Those of you who attended the meeting of the Mathematical Association here in Cambridge, in December, 1922, may recall Professor Cajori's account of the trials and troubles which this idea of limit, as used in the derivative, went through before it reached its present form. It is not strange if this journey has left traces in nomenclature.

Mathematics grows a little at a time. It pleases us to call it logical—and as subjects go it does pretty well in that way—but often it does not proceed like a syllogism.

Take a small example. How do you solve a geometry original? Do you begin at the hypothesis, make deductions from that straight through the first time in complete sequence? Possibly. Do you look at the conclusion—either guessed at or found by somebody else—then look back step by step, seeing what previous statement is needed until you reach the hypothesis? Perhaps. But I think more likely you work out from the hypothesis and back from the conclusion, with maybe a little trying in between, until the ends join up. Then you go over the whole, very likely alter your plan considerably in the light of what you have discovered, smooth up the details, covering your tracks skillfully, and then you sit back and say, "Behold! Isn't it pretty?"

I suspect that something of the sort is the route of a great deal of mathematical invention. So perhaps then it isn't strange that as it has grown mathematics has gathered some inexact symbolisms and terminologies.

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[1] Some additional examples of unfortunate or ambiguous notations which were brought out by the subsequent discussion are: the exponential way of writing inverse trigonometric functions, as $\sin^{-1}x$; $\log^2 x$; $\sqrt{x}$ (indefinite as to sign); and tangent, used in two senses, as a line touching a curve, and as a trigonometric function; especially confusing in the phrase defining the slope of a curve, "the tangent of the angle made by the tangent, etc."

We can truly say, we are improving. We are improving in the clearness of our mathematical language. Let me quote a sample of how it was done once. I came across this proposition in an edition of Euclid's Elements published in 1810. Book V Proposition I.

"If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other."

It is perfectly correct and definite, of course, but I really think we would do better than that nowadays in form of expression.

We are improving in the accuracy and preciseness of our ideas. For instance, referring again to his matter of limits, it is not so very long ago that some of our books defined limit as "a value that a variable approaches but never reaches." My recollection is that it was almost as bad as that in the geometry I first studied. It never troubled me then. "You know what I mean" took care of it all right.

We are improving constantly in our notation. For instance, in the symbolism for limits, too, the arrow, $x \rightarrow a$, instead of the old $x = a$, or $x = a$, is a distinct improvement and deserves universal adoption.

The round d for partial derivatives, is a great advance over using the ordinary d in two senses. And there are others.

Such concerted efforts as the Report of the National Committee are a great move towards definiteness of meaning. That they had to leave a few beautiful straddles only emphasizes the difficulties of the task.

Is there anything we can do about it? Not a great deal, but something. We can welcome and use real improvements when they appear. We can watch ourselves to beware of assuming that of course our students know what we mean, however we express it, and we can frankly confess the pitfalls in our mathematical system where they exist, warn our students of them, and be as patient with them as we can when they tumble in. Perhaps they have a reason, if not an excuse.

It is possible that what I have tried to say has not been perfectly clear. If not, perhaps I, too, can take refuge in "Well. anyway, you know what I mean."

SOME VALUES OF ALGEBRA

HARRY C. BARBER

I

The beginnings of algebra are hard. There are more difficulties of technique than we are wont to admit. Reeve[1] says, "Results show that, for some pupils, the solution of relatively simple equations is more difficult than is often recognized by high-school teachers." See also Thorndike's Psychology of Algebra and other studies of the results of algebra teaching.

A still deeper reason for the initial difficulties of the study of algebra is philosophic, and arises from questionings about why we study algebra, the nature of algebra, the extension of our number system, the important generalizations in the study of symbols, equations, and the function idea, and the mathematical mode of thought as distinct from the historical, scientific, artistic, or philosophical.

These are very real and very great difficulties; too much to overcome in one battle or in one short campaign. The human mind has met and mastered them one at a time at different places and at widely different times.

II

What values do we want to get from our struggles with these difficulties of algebra We, who believe that mathematics has values for nearly every pupil of the junior high school years, want to find these values and state them and learn how best to realize them.

If we want to make computers we should do as a statistical organization might do in training its employees to use the necessary technique. This would be the easiest aim and would be best accomplished by waiting until the pupils were fairly mature, grades eleven and twelve perhaps, and setting up the necessary drills and tests to establish the required technique.

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[1] A Diagnostic Study of the Teaching Problems in High School Mathematics; Reeve, Ginn, 1926.

If what we want to do is to equip our pupils for the mathematical operations of the science class, we should search out the required operations and again evolve the necessary technique for their mastery. Fairly scientific means of accomplishing either of these results could be, probably are being, set up. They would, however, satisfy no teacher except the veriest crammer. Most of us would want to get a little more meat out of the process; a little more educational value; to build up a little more understanding and more ability to cope with emergencies. We would not be content to make algebra what Whitehead calls a dodge for doing this and a dodge for doing that.

In just so far as we are committed to the broader aim, we put the emphasis upon understanding, and see in each new process not so much something to do, as something to think about; an exercise in understanding the mathematical mode of thought. (Is it not likely to be true, in school as well as out, that we spend too much of our time rushing about doing something instead of really thinking?) The method of teaching for understanding is the method of reclarification, by which the pupil is called upon at intervals to re-understand and to think his way out again, and to re-explain. Drills change, too, from the abstract type which characterized the old algebra to repeated practice in the *use* of the process in new work, a procedure which results in a quite different frame of mind.

If what we want is to use algebra as a sieve for screening out college material we can do so, but if we want to use it as a test of the tutor's skill at easing the unfit over the college entrance examination we must admit our willingness to debauch our subject.

If, on the other hand, we have a more philosophic point of view and want to accomplish much bigger things; if we want to use the kind of difficulties algebra presents in order to introduce the pupil to mathematics as one of the great regions of human thought,[1] as a kind of knowledge about which he cannot afford to be uninformed, if we want to give him a real mental voyage of discovery, then we shall think of algebra as a mode of thought

[1] For this phrase and several others in this paper I am indebted to a recent interview with Henry W. Holmes, Dean of the Harvard Graduate School of Education.

instead of merely a technique of manipulation, and we shall regard each one of the difficulties as a possible opportunity for a valuable mental experience, esteeming of less value a computational facility established by a routine, than a habit of thinking over the matter.

III

The meanings of the minus sign.

Let us take one of the difficulties we have been considering, that for instance, of extending our number system to include negative numbers, and see how we may get from its study some of the bigger values.

First we recognize that it is a big idea, a generalization which at one step doubles our number system and complicates the rules for computing with it. (In fact this extension of "numbers" to include negative number changes our arithmetic of positive integers into an algebra.) As such we must take it bit by bit instead of at one gulp washed down with a great amount of explanation and illustration. (Is not this desire to exhaust the exposition of a big topic on its first study one of the best evidences of the amateurishness we are trying to overcome?)

Second we shall postpone the idea and any of its parts until needed in the main line of our progress, that is, in the study and use of formulas, equations, and graphs.

Third we shall distinguish clearly among the three meanings of the minus sign, letting each meaning serve its own purpose. In our first formulas and equations we need only the familiar meaning, subtraction. This cares for such cases as $3 - (a + b)$, $3x - 2x + 7x$, and so forth. When we come to $3 - 8$ we need the second meaning. The result is "5 in the hole," a *shortage* of five, or -5. The minus sign now indicates a suspended subtraction which we are unable as yet to perform, a condition which might unfortunately happen to one's bank account. It is quite familiar to most children from scoring their games or other illustrations. It is an idea which a merchant might use in marking a bale of silk which was found to be of short weight. The world got on for a long, long time with only these two meanings of the minus sign. Let us call the second a *shortage*, in order to give definiteness to the concept, and then see what it explains.

It explains the negative answer to many kinds of problems. Together with the original meaning, subtraction, it explains all the operations with "signed" numbers. The explanations are mostly obvious and do away with any mystery in the mind of the child. (This last is of more importance than we often realize because keeping school work real in the mind of the pupil is one of the most serious problems of modern education.)

There are only two cases requiring any comment here. In $5 - (3x - 2)$ when we subtract the $3x$ we have subtracted 2 more than we were told to subtract and hence must put back the 2. This leads later to $5 - (-2)$ and the observation that subtracting a shortage is the same as adding. In finding the product of a positive and a negative number we tacitly use the commutative law, if necessary, to make the multiplier always positive. In finding the product of two negative numbers the pupil should first meet $-3\ (x - 2)$ in which he first multiplies, and then subtracts as a separate operation. In multiplying -3 by -5 in the vertical arrangement, which may come much later in the year, he continues to multiply and then to subtract until his mind is thoroughly ripe for the mechanical rule. The process is easily enough rationalized if we give to the minus sign before the multiplier its original meaning, that of subtraction.

It is true, as Prof. Ralph Beatley pointed out in the January issue of this journal, that the illustrations of negative number often given by teachers will not bear the strain of explaining all the processes. They should not be expected to do so, even if they are as good as the one I got from Mr. Vosburgh; about the town which lost three paupers who had been costing it $200 a year apiece. It is also true that all of the operations can easily be understood by the use of the *first two* meanings of the minus sign.

The third meaning of the minus sign is geometric and has to do with direction. Compared with the other two meanings it is recent. It is not needed to explain the operations with signed numbers. Trying so to use it has possibly been a cause of confusion. It is valuable in interpreting the answers to some problems. It leads us into the field of analytic geometry.

Few would question that recent changes in algebra teaching have increased its educational value. Getting formulas and equations to the fore so that pupils learn why they need to manipulate before learning rules for manipulation, is important progress. Further advance in getting the introduction to signed numbers, that is, the extension of our number system, distributed over the long period of time which its significance warrants, is probably hindered by the fact that we have forgotten that we can teach the solution of equations without preliminary rules for signed numbers, provided that we return to the rule of Diophantus and "make up the shortages," the step from which the name algebra was derived.

IV

If we think of the algebra course as the gradual unfolding of certain great ideas which enrich the experience of the pupil; if we believe that the most important thing we can do in the algebra class is to get the pupil to think; if we consider it at least as important to have interesting ideas about negative number and its history, as to have rules for manipulation, we may act in the light of the faith that is in us.

Just as we study an epoch of history without expecting to remember all about it, but for the value of the experience, for the frame of mind it gives us and the knowledge of how the human race works out its problems; so we may take young people through the beginning course in algebra for the experience it gives them, for the attitudes it creates, and for the knowledge of that side of human experience which we call mathematics.

DRAWING FOR TEACHERS OF SOLID GEOMETRY

By JOHN W. BRADSHAW
University of Michigan, Ann Arbor, Mich.

*Part Three

In the last article we drew in clinographic axonometry the picture of a right prism with a rhombus for its base. As a second illustration of this method of making pictures, let us consider a regular pyramid whose base is an equilateral triangle ABC and whose altitude h is equal to the edge of the base s, a figure closely resembling the third in the Board's Document. We shall put the prism in such a position that the altitude will be vertical and we may use the plane of the base as our horizontal XY-plane. Then any pair of perpendicular lines in this plane may be taken as X- and Y-axes, but a particular choice will be useful in solving some interesting problems that we shall notice later. So let us choose our X-axis through A parallel to BC and our Y-axis through C perpendicular to BC (Fig. 16). If G is the inter-

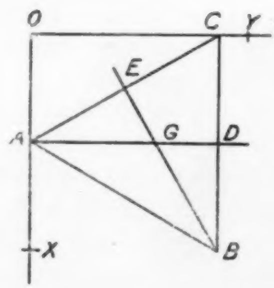

Fig. 16

section of the medians AD and BE of the triangle, this is the point in which the altitude strikes the base. The vertex we shall denote by V, the altitude GV of the pyramid is parallel to the Z- axis.

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*Part One appeared in December, 1924, vol. XVII, pp. 475-481, and Part Two in January, 1925, vol. XVIII, pp. 37-45.

Next choosing our picture-plane coincident with YOZ as we did for Fig. 15, we draw in it two perpendicular lines and on them lay off OY and OZ equal to the unit in space (Fig. 17). The X-axis is drawn down to the left and X^* is marked so that OX^* is something like 2/3 of OY. (We observed that this point could be taken at pleasure.) Into this picture-plane we revolve the XY-plane about OY and obtain an exact reproduction of Fig. 16. In using the notation X_1, A_1, B_1, D_1, G_1, we intend to suggest that we have here not the points in space nor yet their pictures but the positions that the points X, A, B, D, G would occupy after a quadrant revolution about OY.

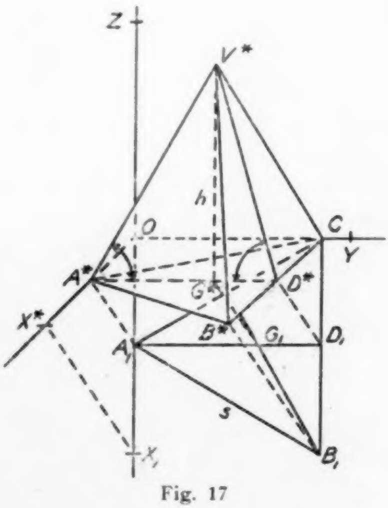

Fig. 17

Since X^* and X_1 are respectively picture and revolved position of the point X in space, the dotted line joining these points is the picture of a quadrant chord and determines the direction of all like pictures. Thus A^* lies on OX^* where it is intersected by a parallel to X^*X_1 through A_1; then A^*D^* must be parallel to OY, since the parallels OY and AD must be pictured by parallels; D^* and G^* are located on this line by drawing through D_1 and G_1 parallels to X^*X_1. To find B^* we draw through B_1 a parallel to X^*X_1 and note its intersection with CD^*. Finally the altitude of the pyramid is drawn in its true length through

G^* perpendicular to the Y-axis, and thus we have V^*, the picture of the vertex. The pictures of the edges are merely the lines joining V^* with A^*, B^*, and C.

Various tests of accuracy are readily obtained, for instance CB^* should be parallel to OX^*, for these lines are pictures of parallel lines in space.

We made our choice of axes having in mind the solution of two geometrical questions. They are: What angle does an edge make with the base? And what is the diedral angle formed by one of the faces and the base? Both of these angles can be measured in the plane ADV, and since this plane is parallel to the picture plane, figures in it appear in their true form in the picture. The two angles in question are actually $\angle G^*A^*V^*$ and $\angle G^*D^*V^*$, angles which we can measure in the figure with a protractor. They are about 60° and 74°, respectively.

The Board suggests the use of trigonometry in solving such problems. A comparison of the results obtained graphically by measurement with those obtained by calculation will do much to increase the pupil's assurance. Thus from trigonometry we find answers to the questions proposed. Since $GV = h = s$ and

$AG = \frac{2}{3} AD = \frac{2}{3} \frac{\sqrt{3}}{2} s$, we have

$$\tan \angle GAV = \sqrt{3}, \qquad \angle GAV = 60^\circ$$

And similarly

$$\tan \angle GDV = 2\sqrt{3}, \qquad \angle GDV \quad 73^\circ\ 54'$$

The objection may be raised that Fig. 15 and Fig. 17 are complicated. That is to be granted, they are not very simple. The reason is that they contain all the machinery exposed to view, all the construction lines are there. In the finished picture all that we need to show how we worked, aside from the space figure itself, is that little triangle OX^*X_1; everything else is derived from it.

The great merit of our procedure lies in its reversibility, in our ability to infer the space figure from a picture. Two problems of quite distinct types will bring this out.

In the first we have given the picture $A^*B^*C^*$ of a triangle in the XY-plane, together with the triangle OX^*X_1 showing how it was made (Fig. 18), the problem is to find the true form of the triangle ABC. Fig. 19 contains the solution of the problem.

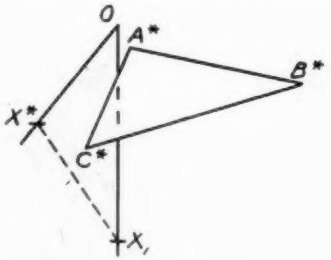

Fig. 18

Through C^* we draw a parallel to the Y-axis cutting OX^* in U^* and through B^* a parallel to OX^* cutting OY in W. If these two lines intersect in V^*, the parallelogram OU^*V^*W is clearly the picture of a rectangle, which when revolved about the Y-axis into coincidence with the picture-plane appears as $OU_1\ V_1\ W$, U_1 and V_1 being determined by the condition $U^*U_1 /\!/ V^*V_1 /\!/ X^*X_1$. On the sides of this rectangle lie C_1 *and* B_1 found by considering that B^*B_1 and C^*C_1 are pictures of

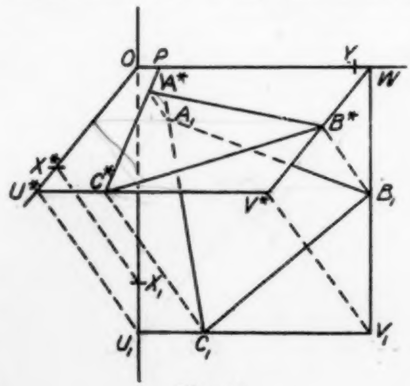

Fig. 19

quadrant chords and must be parallel to X^*X_1. To determine the revolved position of A, produce C^*A^* to intersect OY in P. Then C_1A_1 must pass through this *same* point and $A^*A_1 /\!/ X^*X_1$.

The triangle $A_1B_1C_1$ reveals the true form of the triangle pictured. We observe by measurement that the sides are, as near as we can tell, equal. We may say then that in a sense the triangle OX^*X_1 replaces the description, "ABC is an equilateral triangle."

The other problem involves the determination of OX^*X_1 from a description. Here (Fig. 20) the data are the picture A^*B^*CD taken from Fig. 5 (Part I) and the description, "the figure is a rhombus with 60° angle at A lying in the XY-plane." Our agreement about notation indicates that CD lies in the picture-plane and may be taken as Y-axis with origin at D. We then construct the rhombus in true form DCB_1A_1, thinking of it as revolved into the picture-plane. If M_1 is the point in which the revolved X-axis, perpendicular to DC, cuts A_1B_1, and if M^* is the picture of M on AB obtained from the fact that $M_1M^* \parallel A_1A^*$ then DM^*M_1 may serve as our triangle OX^*X_1 to give us the direction of projection.

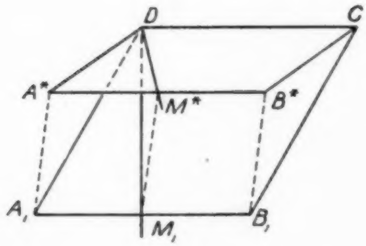

Fig. 20

The amount of freedom which we have found in making parallel projections may almost seem to render our whole discussion useless. If any parallelogram will represent a 60° rhombus or a square at pleasure, if any triangle is a good picture of an equilateral triangle from some direction of vision, what's the use of having any rules or fixed and complicated procedure for making pictures? The reason for this seeming indifference lies in the simplicity of the figures considered. The picture in each case has had just enough in it to determine the direction of projection. If we had a cube and a regular triangular pyramid standing side by side and wanted to put them in the same picture we could not be so free. We might choose any parallelogram to represent the square base of the cube, but then the picture of the base of the pyramid would be determined, or vice versa. If we chose both

at pleasure our picture would contain inconsistencies and might lead to altogether false conclusions.

Examples in point may be found in the Board's Document. Each of the first five figures shows a solid and right beside it a part of the same solid. In four of the five the direction of projection for the two pictures is not the same, and hence they must be considered as distinct pictures rather than as parts of the same picture. As illustration we are reproducing, with slight alteration in irrelevant particulars, Fig. 4 of the Document and are calling it Fig. 21. Part (a) of this figure represents a frus-

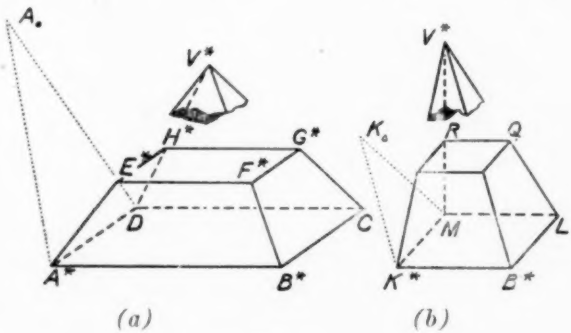

Fig. 21

tum of a regular pyramid with a square base, and part (b) shows a quarter of the same solid. Although both of these figures are correct pictures of the solids described, in that the altitudes of the pyramids are the same, as are also the altitudes of the frustums, and K^*B^* is one-half of A^*B^*, yet on account of the difference in the direction of projection the picture of the frustum of the quarter pyramid gives the impression of greater height. The triangle DA^*A_0 is drawn to show that projecting rays in part (a) make an angle of about 66° with the picture-plane, while in part (b) this angle is about 58°, as shown by MK^*K_0.

The prism and pyramids that we have been talking about thus far were easy to describe because we had quite specific names by which to designate them, the prism was "right" and the pyramids were "regular." In effect this enabled us to consider the problem as chiefly one of picturing the base; we were concerned with representing a figure lying in the XY-plane. But in the

problems suggested by the Board mensuration of pyramids is not limited to regular pyramids and the frustums of regular pyramids. Problem 7 has a figure of its own here reproduced as Fig. 22 and the questions relate to the pyramid whose vertex is V and whose base is $ABCD$. This is described in the statement, "$ABCDA'B'C'D'$ is a cube of edge 12″ and V is the mid-point

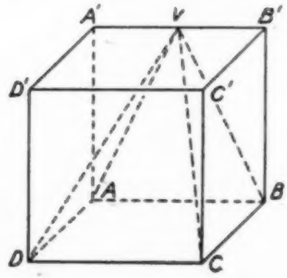

Fig. 22

of $A'B'$." The picture of the pyramid alone, even with a set of axes inserted, would not enable us to dispense with a description and we are here confronted with some fundamental considerations that require further study. What conventions can be made so that a picture will completely characterize an irregular object?

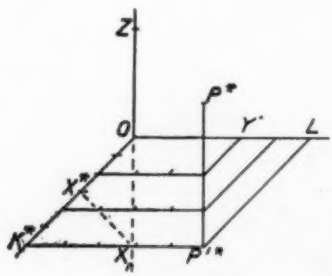

Fig 23

To answer this question we must return to our determination of the position of a point in space by means of co-ordinates. A set of axes in the drawing and the picture P^* of a point do not fix the point in space. Thus the point pictured in Fig. 23 may

be equally well the point (2, 3, 2) or (4, 4, 3) or (6, 5, 4) or any one of an unlimited number of other points. In fact it is clear that all the points along a projecting ray must have the same picture. If, however, we agree that the broken line OLP'^*P^* or $OK^*P'^*P^*$ shall indicate how P is reached from the origin along the axes and parallels to them, we can tell what co-ordinates must be associated with P and we have a unique determination. But more simply still, it is not necessary to draw the whole broken line, all we need is P'^*. This is the picture of the foot of the perpendicular from P on the XY-plane, we might call it the axonometric XY-projection of P. It lies, of course, on the parallel to the Z-axis through P^*. *A point in space is determined by its picture and its axonometric XY-projection.* We commonly draw a line of short dashes to connect these two points. When no axonometric XY-projection of a point is given it will be understood that the point lies in the XY-plane. With these conventions Fig. 24 would be recognized as the picture of the pyramid of Fig. 22 without the aid of the cube or of any description.

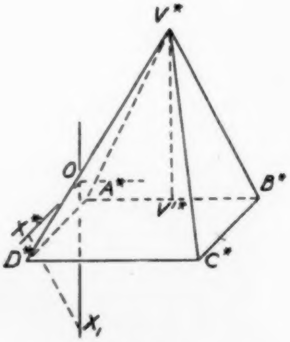

Fig. 24

Finally, lest we should seem to claim too much for our pictures, it should be clearly stated that their definiteness rests upon the tacit assumption that the solids represented are bounded by plane faces. The picturing of simple curved surfaces, such as those included in the Board's Document, will be taken up in the next article.

AN ANALYSIS OF FRESHMAN COLLEGE MATHEMATICS

By PROF. E. E. WATSON
Iowa State Teachers' College

The question is frequently raised as to why college freshmen find mathematics difficult. In order to discuss this intelligently let us examine not only the course itself but the amount and nature of the material in geometry that is assumed as a foundation for freshman college mathematics. In elementary geometry we find certain features such as:

1. The vocabulary
2. The basic ideas
3. The theorems

The expression "basic idea" is here used to mean a fact or group of elementary facts, a knowledge of which is necessary for a definite concept of the theorem under consideration. Thus the statements:

"A point has only position."
"A line-segment is any part of a line."
"A triangle has three sides."
"The sum of the acute angles of a right triangle is 90°."

are called basic ideas. In the latter statement it is necessary to know what acute angles are, what a right angle is, what a triangle is, and that the sum of all the angles is 180°, a group of elementary facts.

An analysis of recent texts on high school geometry indicates that such a book contains:

1. A semi-technical vocabulary of 600 words, such as abscissa, arc, complement, diagonal, function, initial, locus, maxima, obtuse, projection, right, sine, trisect, variable, =, etc.

2. Two hundred seventy-five basic ideas such as:
    1. A minor arc is the smaller of two unequal arcs of a circle,
    2. Concentric circles have the same center,
    3. There are 360° in a circle.

3. Seventy theorems which most geometries contain and which for convenience are here called "essential theorems."

4. Forty theorems in which there is considerable variation in the selection.

Of the above named material what part is used either directly or indirectly in freshman college mathematics during:

(a) The first quarter, three months.

(b) The second quarter, that is not used during the first quarter.

(c) The third quarter, that is not used during either of the other quarters.

(d) How much new material is used each quarter?

An analysis of the first quarters work. A careful tabulation indicates that the college freshman taking a course in unified mathematics at the Iowa State Teachers College uses of the above named material during the first three months:

(a) 320 semi-technical words or 53% of them.

(b) 154 of the basic ideas or 56% of them.

(c) 51 of the essential theorems or 73% of them.

In addition to the above named material the student must use:

(a) 600 semi-technical words, not found in geometry such as, acceleration, bearing, co-function, dip, exponential, graph, inclined, latus rectum, moment, radian, synthetic, x-axis, etc.

(b) Seven theorems, not among the essentials.

(c) Formulas and processes used in connection with the topics; graphs, numerical trigonometry, logarithms, straight line, quadratic equation, slope, maxima, theory of equations.

An analysis of the second quarter's work. The second quarter's work is so organized that by the end of this term the student has completed his college algebra, trigonometry, the straight line formulas of analytic geometry and has had both differentiation and integration. The term's work may be summarized as follows:

(a) Semi-technical words used in geometry, but not used in freshman mathematics during the first quarter 32

(b) Semi-technical words not used in geometry or in the first quarter's work 500

(c) Basic ideas not used in first quarter 14

(d) Essential theorems not used in first quarter 14

(e) Theorems not among the essential theorems........ 8

(f) Formulas and processes of the quarter's work.

An analysis of the third quarter's work. During the third quarter the analytic geometry is completed. Some attention is given to complex numbers, polar co-ordinates, statistics, differentiation and integration. This term's work may be summarized as follows:

(a) Semi-technical words not used in the work of the first or second quarter..300

(b) Theorems of geometry which deal with the cube, rectangular solid, cones, spheres, parabolas, ellipses, hyperbolas.

(c) Formulas and processes of the course.

By the time the student has completed the work of the third quarter he has had occasion to use, either directly or indirectly:

(a) 350 of the semi-technical words of plane geometry.

(b) 168 of the basic ideas.

(c) 65 of the essential theorems.

(d) 15 theorems, which are not among the essentials.

(e) 1400 new semi-technical words.

Of the 70 essential theorems the following are not used or only rarely occur in this course.

1. Intersection of chords within a circle.
2. Perimeters of two similar polygons.
3. Test for similarity of polygons.
4. The sum of the exterior angles of a polygon.
5. The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.
6. The bisector of an exterior angle of a triangle divides the opposite side externally into segments proportional to the adjacent sides.
7. An angle formed by two secants, two tangents or a tangent and a secant equals in degrees $\frac{1}{2}$ of the difference of the intercepted arcs.

The theorms here called "essential theorems" correspond very closely to the ones recommended by the national committee.

From the above data it is evident that a fair knowledge of plane geometry—vocabulary, basic ideas, theorems—is one of the essentials to success in freshman college mathematics.

SUGGESTIONS ON CONDUCTING THE RECITATION IN GEOMETRY

By JASPER O. HASSLER
Professor of Mathematics
University of Oklahoma

It is not the purpose of this article to give detailed directions concerning what should be done first, second, third, etc., during a recitation period, nor how much time should be given to each division of the recitation. Giving such advice is too much like selling a "patent medicine." A fixed recitation plan can not be used by all teachers with any more assurance than can a patent medicine be expected to cure all the diseases listed on the label. To give advice in such a manner is equivalent to suggesting limitations on a teacher's independence and originality. What is done at any particular time during a recitation depends on the nature of the subject matter in that day's lesson, the nature of the class, and the methods of the teacher. The recitation program may not always be chronologically the same. A wise teacher soon learns to do the right thing at the right time. It is sometimes necessary to break away from an outlined recitation plan for the sake of making a point—to step aside from the beaten path and meet a situation unexpectedly arising which, because of the manifest interest of the class, furnishes rare opportunity to "drive home" the principles of the subject.

It is the purpose of this article to deal in a general way with methods of presenting the subject matter of geometry in such a manner as to develop powers and habits of careful, accurate, and independent thinking rather than to present geometry as a finished model of deductive logic.

Teaching the Proved Propositions

In a certain geometry class a girl was reading on the blackboard her written proof of a proposition when a pupil asked, "Why is Axiom 5 used as a reason for that statement?" The teacher indicated that the girl should answer. "I do not know why," she said, and then added with an air of finality, "but I

know it is by Axiom 5.'' That girl should have been trained to expect to do something during the recitation period which would test her understanding of the theorem and the proof of it rather than merely to be able to write down the statements of the book.

There are so many better and briefer ways to measure a student's comprehension of a theorem that it seems a pity for so many teachers to cling to the timeworn method of having him write out the proof on the blackboard as given in the text. It is easy to devise questions about the theorem and its proof, the answers to which will reveal more quickly and accurately the student's insight into the proof than will the process of writing down the successive steps of a proof that he may have memorized. Illustrations of other methods may be found in the following paragraphs.

Suppose a class has just studied the theorem, "The sum of the interior angles of a triangle is equal to two right angles." Let us assume that the method of proof in the book used a figure

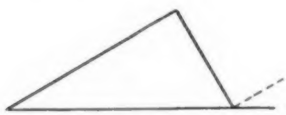

Fig. 1

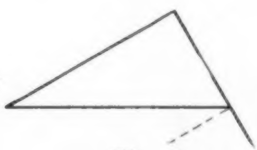

Fig. 2

like Fig. 1. The class should be required to write the proof in recitation with a figure like Fig. 2, or with some other similar variation of the figure. A more difficult test would be to give the figure of the triangle with the auxiliary line through the

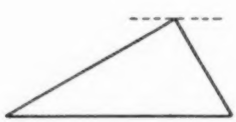

Fig. 3

vertex parallel to the base as in Fig. 3. The student who can not complete the proof with simple changes of the figure has really failed to comprehend the proof as he should. The better stu-

dents of the class should be able to prove the theorem by the use of both variations of the figure as suggested above. Less than half the class period will be required for them to write the proofs and all have recited instead of a few at the blackboard. The remainder of the period can be spent in encouraging thought by developing in a heuristic manner the theorems ahead, or used for the purpose of explaining parts of the lesson for the day not clearly understood. If the pupils learn to expect this kind of a recitation instead of the too frequent order to a few, "Go to the board and put on Proposition X" (or some other number), they will do something else besides memorize, without thinking about them, the statements in the book. It can not be done that way on every theorem and need not be done every day, but frequent recitations like that will not only inspire in the pupils greater effort to understand the matter in the text, but will tell a teacher more about his pupils' understanding of the proofs than anything else, except possibly the verbal replies to questions in class.

Sometimes it is easy to parallel the statements of the theorem by a concrete numerical example which may be given for the pupils to consider instead of the abstract statements of the theorem itself. For example, suppose the theorem is, "The bi-

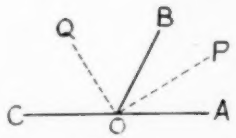

Fig. 4

sectors of two supplementary adjacent angles are perpendicular to each other." For the recitation (to be written by all) the following fact may be assigned to be proved by deduction. The line OB (Fig. 4) makes an angle of 66° with the line AC. OP and OQ bisect the angles AOB and BOC. Prove (as a theorem) that the angle POQ is a right angle. The successive steps of the proof would be: $\angle BOC = (180° - 66°) = 114°$; $\angle BOQ = 57°$; $\angle BOP = 33°$; $\angle POQ = (57° + 33°) = 90°$. The general proof has parallel statements.

As another illustration, consider the theorem, "The perimeters of two similar polygons have the same ratio as two homologous

sides." The following written exercise may be assigned as that part of the recitation which seeks to discover whether or not the proof of the theorem is understood. "Given the polygon ABCDE with the sides, $AB = 4$, $BC = 6$, $CD = 10$, $DE = 8$, $EA = 12$; a similar polygon, $A' B' C' D' E'$ has the side $A' B' = 6$; prove that the ratio of the perimeter of ABCDE to the perimeter of $A' B' C' D' E'$ is 2/3.

A wise teacher will sometimes use concrete, numerical examples like those suggested in the two preceding paragraphs in the introduction of the theorems to the class before assigning them, making use of the inductive method. Otherwise they may be used as suggested above.

Let it not be understood that my purpose is in anyway to minimize the value of the model proofs of the theorems as usually given in a text or the knowledge of those proofs by the pupil. The purpose is to suggest methods of recitation that will emphasize the comprehension of the proof rather than encourage the memorization of the proof in the text. A part of the training in geometry (but not so great a part as some teachers make it) should certainly be for the purpose of enabling the pupils to set down in logical order the steps in the proofs of the fundamental theorems.

Efficiency in Provoking Thought

A proper method of measuring a teacher's efficiency in securing mental activity on the part of the pupils would be something like the following. Let us assume there are thirty students in a class and that the effective recitation period (eliminating time for classroom changing) is forty minutes. There are 1200 "student-minutes" of thinking possible if thirty pupils are engaged during the entire period in productive thought. Suppose the pupils spend on the average about thirty minutes of this time actually thinking about geometry. This makes 900 "student-minutes" of thinking. The efficiency is .75.

The question immediately arises, "How can this efficiency be increased?" It certainly does not tend to increase the number of student-minutes of thinking for half the members of the class to lounge in their seats while waiting for the other half to write out proofs of propositions on the blackboard. The seated

students should be working on some original exercise while waiting. It does not increase very much the number of student-minutes of serious thought for twenty-nine students to sit and listen to one student read what he has written on the board as they attempt also to follow the reading which in many cases, because of lack of space for writing, is a difficult task in itself. Better teachers have only the figures drawn on the board, the remainder of the class being occupied with some new work during this process. Then, with all at attention, each student gives orally the proof of the theorem assigned to him, the class being kept on the alert by knowing from past experience that any one of them at any time may be called upon to "take the words out of the mouth of the speaker" in giving a reason or the next statement. The students understand that their success or failure in their responses to these "snap questions" has a decided bearing on what the teacher thinks of their ability.

Some teachers have been successful in avoiding loss of time during roll-call, etc., by having an exercise for the class written on the board before the class comes into the room. Each student is to take his seat and begin work as soon as he enters the room. This method has the double advantage of increasing efficiency and avoiding much disorder that frequently attends the assembling of a class. The exercise may be a numerical one designed to test for an appreciation of the fact that the abstract statement about geometric relation as given in the theorem applies to concrete things. It may be an easy "original theorem" or a new construction based on the theorem of the day. It may be a preliminary exercise introducing a line of inductive thought leading to the development of the proposition for the next lesson. Frequently the teacher can, by passing among the pupils as they work, make sufficient observation of the nature of each individual's work that it will not be necessary to take up the papers.

In the matter of provoking thought, no device can supplant a heuristic attitude on the part of the teacher. His bearing in the classroom should radiate questions and should inspire the pupils to do their best to answer them. The pupils should be made to feel that the business of the teacher is not so much to answer their questions as to lead them to find the answers. Suppose, for example, that a student attempts to prove that "the

sum of the interior angles of a triangle is equal to two right angles." He has a mental picture of a figure like Fig. 5 but does not know how to draw the auxiliary line through B. He has *forgotten* that it should be parallel to AC. The following conversation might occur.

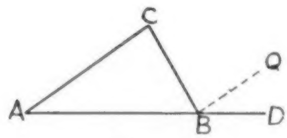

Fig. 5

Pupil: I do not know how to draw the line, BQ.

Teacher: (Instead of telling him it should be parallel to AC) For what purpose do you draw the line?

P. In order to get equal angles.

T. To get what angles equal?

P. To get the angles that BQ makes in the angle CBD equal to the angles A and C.

T. Are the angles C and QBC situated with respect to each other in any manner that reminds you of a theorem about equal angles?

P. They look like alternate-interior angles.

T. Under what condition are such angles equal?

P. A transversal must cut two parallel lines.

T. How then would you draw BQ?

P. Parallel to AC.

Every teacher should cultivate the heuristic attitude and learn how to ask the right kind of questions.

Measuring Progress

A teacher's estimate of a pupil's ability in geometry or any other course in mathematics should not be confined to the average of a series of numerical grades across the page of class book. It should not depend too much on the grade made in a final examination, although that certainly must have weight. There will necessarily be many grades recorded. Some means of obtaining expressions from the students of the class, as a basis for grading.

have been suggested on preceding pages; for example, solving a numerical problem, proving an assigned proposition a new way, or proving an original exercise. These grades furnish a good index of a pupil's ability; but a skillful teacher can discover very much more by the intelligent nature of the replies of the student to his questions day after day. What teacher of experience has not seen with a thrill the growth of initiative, independence of thought, reasoning ability, increased confidence in his own powers, ability to make application of the subject matter, and (with it all) a constantly growing interest in the subject, by pupil after pupil in his class! If a teacher has conscientiously kept close to the thoughts of his class in geometry, he needs no average of numerical grades nor final examination to determine their standing (with some exceptions) unless it be merely to confirm his estimates and assure him that his grades are fair. He should certainly be able to discover more about a pupil in four or five months of daily conversation, consisting of questions and answers, than can be discovered in the two or three hours alloted to an examination. This continued appraisal of a student is part of the teacher's work in every recitation.

Let this additional word be said about one phase of an examination, or written test, or "quiz." If the students are to be tested on their ability to prove original exercises, the test should not include any of the exercises studied in the text. We should choose instead, some exercises which the class has never seen. If a limited time is allowed for the test, the exercises must not be very difficult, nor should the construction of the necessary figures require a great deal of time. I do not mean to imply that the pupils should not be tested on their ability to prove the fundamental theorems and various minor theorems or exercises having important practical applications, but this is too often a test of memory rather than a test of initiative in the use of geometric facts to establish other facts.

Teaching Scientific English

So much modern reading is for thrills instead of thought that a student needs to be taught how to read and express statements that require thought. A text in mathematics can not be scanned

at a glance like a typical "continued on page 148" magazine story where the reader can easily trace the run of the plot and scan over a great deal of "padding" put in by the writer. There are no superfluous adjectives in scientific English. A reader must go slow and think. The pupils can learn to appreciate the beauty and simplicity of this type of writing by learning to write and talk that way. They should learn to *say what they mean.*

One way to encourage care and accuracy in making statements is to have a considerable amount of work done at the board at the dictation of individual students, using a plan of procedure much like the following. Let us assume that the teacher is asking the pupil to suggest the necessary figure in order to develop some theorem about a tangent. A circle is drawn, a radius is drawn, and the student says, "Now draw a line perpendicular to the radius." The teacher does so, at some interior point. "No," exclaims the student, "I mean at the end of the radius." The teacher then draws the perpendicular through the center of the circle and the pupil is taught that he should have said at the beginning that the perpendicular be drawn at the point of intersection of the radius with the circle. If managed so as not to emphasize too much the ludicrous situations that will naturally arise and cause a loss of teaching value, this method of doing something not intended by the pupil but which may fulfill his instructions can be made of great value in producing definite and accurate statements. There are other schemes which accomplish the same purpose. Some teachers have done a similar thing by drawing a figure fulfilling the definition given by a pupil but which is not the figure intended by the pupil. Indeed, it is sometimes possible to follow the instructions given by the definitions in the text and draw figures that would surprise the author.

It is not my purpose to emphasize any particular scheme, but to make clear that it is the duty of the teacher of geometry to insist on accurate statements of definitions and theorems (not necessarily memorized from the book) and thereby assist in the teaching of English. Indeed, the teacher of every subject in the curriculum should do his bit in the teaching of English.

BOOKS THAT HELP MAKE MATHEMATICS INTERESTING

By TRUMAN P. SHARWELL
Bogota High School, Bogota, N. J.

I. Books for Mathematics Clubs

Abbott, E. A. *Flatland: A Romance of Many Dimensions, by* A Square (E. A. Abbott). Little, Brown and Co., Boston. 155 pp. $1.25.

A charmingly written popular treatment of that much discussed idea, the fourth dimension.

Archibald, R. C. *Mathematics and Music.* The Argosy, August, 1924, pp. 135-142, Sackville, N. B., Canada. $.25.

Batchelder, P. M. *The Place of Mathematics.* University of Texas Bulletin, No. 2420, May 22, 1924, pub. by The University of Texas, Austin.

Mathematics as the final language of truth. A reading of this inspiring article gives one a better understanding of the power of mathematics and mathematical methods.

Black, H. G. *Paths to Success.* D. C. Heath and Co., New York, 1924.

A series of 16 essays on secondary school subjects specially written for pupils by eminent educators of America. The one on Mathematics (pp. 114-135) is by Prof. George William Meyers, Univ. of Chicago. Titles of paragraphs: (1) What Is Mathematics? (2) The Modern Meaning. (3) The Start and First Acquaintance. (4) The Story of the Christening. (5) Geometry, the Earth-Measurer. (6) Algebra, the Arabic Name. (7) Trigonometry, the Triangle-Measurer. (8) Who Have Studied Mathematics? (9) The Beginning in Europe. (10) Mathematics Does Not Outgrow Itself. (11) How Mathematics Got Started. (12) What Keeps Mathematics Going? (13) Mathematics for the Love of it. (14) Modern Reasons for Mathematics. (15) Eliot's Six Essential Constituents of Education. (16) Butler's Evidences of Culture. (17) What Mathematical Study Will Teach. (18) An Ancient Classification of Men. (19) Applied Mathematics. (20) Mathematics as a Helper. (21) New Uses of Mathematics. (22) Mathematics and the Larger Life. (23) Precise and Definite Thinking. (24) Mathematics, the Final Language of Truth. (25) Grading a Class. (26) Mathematized Thinkers in Demand. (27) Mathematized and Partizan Thinking. (28) Mathematics and City Government. (29) Machinery of Mathematized Thinking. (30) Mathematics and Democracy.

BOON, FREDERICK CHARLES. *A Companion to Elementary School Mathematics.* Longmans, Green and Co., London and New York, 1924.

CLARK, NEWLIN, AND SMOTHERS. *The Adventures of X.* D. C. Heath and Co., New York. $.48.

An illustrated play dealing with The Land of Algebra. Illustrates by characters in costume and by dialogue many of the principles of algebra; is witty and entertaining and suitable for acting.

COLLINS, A. FREDERICK. *Short Cuts in Figures.* Edward J. Clode, New York, 1916.

Chapter IX is on Magic with Figures.

CRAWFORD, ALMA E. *A Little Journey to the Land of Mathematics.* Mathematics Teacher, vol. 17, pp. 336-342 (Oct. 1924).

DUDENEY, HENRY ERNEST. *The Canterbury Puzzles.* Thomas Nelson and Sons, New York. 255 pp. $1.50.

This book contains 114 puzzles with their solutions. Contents: (1) Introduction, (2) The Canterbury Puzzles, (3) Puzzling Times at Solvamhall Castle, (4) The Merry Monks of Riddlewell, (5) The Strange Escape of the King's Jester, (6) The Squier's Christmas Puzzle Party, (7) Adventures of the Puzzle Club, (8) The Professor's Puzzles, (9) Miscellaneous Puzzles, (10) Solutions, (11) Index.

DUDENEY, HENRY ERNEST. *Amusements in Mathematics.* Thomas Nelson and Sons, New York. $1.50.

A large collection of puzzle problems from arithmetic, algebra, and geometry.

HILL, T. *Geometry and Faith.* Lothrop, Lee, and Shepard Co., 275 Congress Street, Boston. $1.25.

By a former President of Harvard University.

HUBBARD, ELBERT. *Hypatia.* (Little Journeys to the Homes of Great Teachers.) The Roycrofters, East Aurora, New York.

The life story of the woman mathematician who was immortalized by Charles Kingsley in his novel "Hypatia."

HUBBARD, ELBERT. *Pythagoras.* (Little Journeys to the Homes of Great Teachers.) The Roycrofters, East Aurora, New York.

Like the preceding, an attractively printed booklet. Tells of this famous Greek geometer, (1) his trip to Egypt, (2) his severe initiation at the hands

of the Egyptians, (3) his journey to Southern Italy where he founded at Crotona the famous Pythagorean school, (4) the powerful influence of the Pythagoreans, (5) the burning of the Pythagorean College, and (5) the murder of many of the Pythagoreans including Pythagoras himself. "You cannot dispose of a great man by killing him."

Illustrated London News, July 11, 1925.

Contains an interesting recent practical application of elementary geometry. By measuring a side of a triangle (several miles long) and two angles, found the height of the Aurora Borealis (Northern Lights) to be 60 miles.

JOHNSON, WILLIS E. *Mathematical Geography.* American Book Co., New York. 336 pp. $1.20.

Contents: I Introductory, II The Form of the Earth, III The Rotation of the Earth, IV Longitude and Time, V Circumnavigation and Time, VI The Earth's Revolution, VII Time and the Calendar, VIII Seasons, IX Tides, X Map Projections, XI The United States Government Land Survey, XII Triangulation in Measurement and Survey, XIII The Earth in Space, XIV Historical Sketch. Appendix. Glossary. Index. Numerous illustrations.

JONES, S. I. *Mathematical Wrinkles.* Published by the author, Life and Casualty Bldg., Nashville, Tenn. $2.10.

A book of puzzles, knotty problems, quotations on mathematics, historical notes, and other recreations in arithmetic, algebra, and geometry, with solutions.

LEACOCK, STEPHEN. *Literary Lapses.* Dodd, Mead, and Co., New York, 1923. 248 pp.

Contains two of the best pieces of mathematical humor ever written: (1) "Boarding-House Geometry," pp. 26 and 27, a delightful parody on the definitions, axioms, postulates, and propositions of geometry. (2) "A, B, and C: The Human Element in Mathematics," pp. 237-245.

LICKS, H. E. *Recreations in Mathematics.* D. Van Nostrand Co., 8 Warren Street, New York. 165 pp. $1.50.

A large number of interesting problems which might be considered trick problems are presented to afford recreation for an idle hour and to arouse the interests of young students in further mathematical inquiries. Contents: Arithmetic, Algebra, Geometry, Trigonometry, Analytic Geometry, Calculus, Astronomy and the Calendar, Mechanics and Physics, Appendix.

LODGE, SIR OLIVER. *Pioneers of Science.* Macmillan Co., New York. 404 pp.

Interesting accounts of the great astronomers. Copernicus, Tychco, Brahe, Kepler, Galileo, Descartes, Newton, and others. The summary of facts on Falling Bodies, pp. 81-83, and the diagram, p. 82, of the curve (parabola) described by a projectile, showing how it drops from the line of fire 16, 64, 144, 256 feet, etc., in successive seconds, are interesting. How the mention of the path of a baseball as an arc of a parabola enlivens the study of the parabola in algebra!

McSorley, Kathryn. ***Mock Trial of B versus A, or Solving a Personal Equation by the Judicial Process.*** School Science and Mathematics, Chicago, vol. 18, no. 7 (October, 1918), pp. 611-621.

Stephen Leacock's story "A, B, and C" put into dialogue form, with some clever additions to the original. Delightful humor.

Newcomb, Simon. ***Side-Lights on Astronomy.*** Harper and Brothers, New York, 1906. 350 pp.

Chapter X entitled "The Fairyland of Geometry," pp. 155-164, is a brief article on the fourth dimension.

Phin, John. ***The Seven Follies of Science.*** D. Van Nostrand Co., New York, 1912. 240 pp. $1.50.

A popular account of the most famous scientific impossibilities and the attempts which have been made to solve them. Contents: Squaring the Circle, Duplication of the Cube, Trisection of an Angle, Perpetual Motion, Transmutation of Metals (Alchemy), The Fixation of Mercury, The Universal Medicine and the Elixir of Life, Additional Follies, Paradoxes, Marvels, Curious Arithmetical Problems, Numerous Popular Fallacies and Common Errors. Examples: The fourth dimension, pp. 117-125; geometric proof that 64 = 65, pp. 126-127; optical illusions, pp. 152-155; the chess-board problem (sum of a geometric progression), pp. 163-164; the woman and the eggs, p. 170; Archimedes and lifting the world with a lever, pp. 171-173; Newton and the soap bubbles, pp. 179-181. Illustrated. Written in popular, non-technical language involving nothing more advanced than "the simple rules of arithmetic and the most elementary propositions of geometry."

Pond, De Witt C. ***Drafting-Room Mathematics: Problems of the Drafting-Room Simply and Clearly Explained for the Draftsman and the Architect.*** Charles Scribner's Sons, New York, 1924.

Interesting as exhibiting how the architectural draftsman makes use of the simpler portions of geometry and trigonometry.

Popular Science Monthly. *Can You Solve the World's Greatest Puzzles?* Popular Science Monthly, 250 Fourth Avenue, New York, May, 1926. $.25.

A second article by Sam Lloyd is to appear in the June, 1926, issue.

Reno, Lionel C. *An X-Ray on Mystery: Rules and Facts.* Federal Educational Institute, 414 Eighth Avenue, New York, $.25.

A 64-page booklet of the inspirational variety.

Rounds, Emma. *The Pursuit of Zero.* Mathematics Teacher, vol. 17 (October, 1924), pp. 365-367.

A bit of humor written by a pupil of the Linccoln School of Teachers College, Columbia. After the manner of Leacock's "A, B, and C."

RUPERT, WILLIAM W. *Famous Geometrical Theorems and Problems with their History.* D. C. Heath and Co., New York. (Heath's Mathematical Monographs,) In four parts. Ten cents each.

SCHOFIELD, A. T. *Another World, or The Fourth Dimension.* Macmillan Co., New York, 1920. 92 pp.

SCIENCE AND INVENTION. *The Greatest Three Figure Number.* Science and Invention, 53 Park Place, New York, vol. 13, no. 12, April, 1926, page 1093.

This is a brief illustrated article—one of the best that have appeared in a long time. With only three figures one can write a number so large that if written out it would contain 369,693,100 digits. If a space of $\frac{1}{16}$ inch were left between figures, it would require to write this number a strip of paper 919 miles long; this paper would stretch from New York almost to Chicago. If a person wrote one figure a second, ten hours a day every day in the year, it would take him 28 years and 48 days to write this tremendous group.

SLOANE, T. O'CONOR. *Rapid Arithmetic.* D. Van Nostrand Co., 8 Warren Street, New York. 198 pp. $1.50.

An exposition of quick and special methods in arithmetical calculation and a short collection of oddities and recreations in the science of numbers.

SMITH, THOMAS. *Euclid: His Life and System.* Charles Scribner's Sons, Fifth Avenue at 48th Street, New York, 1902. (In the series called The World's Epoch-Makers, edited by Oliphant Smeaton.) 227 pp. $2.00.

WENTWORTH AND SMITH. *Solid Geometry.* Ginn and Co., New York.

Eight of the best known puzzles of geometry, pp. 449-452. History of Geometry, pp. 453-457. (Somewhat similar material may be found in D. E. Smith's "Essentials of Plane and Solid Geometry," Ginn and Co., pp. 481-484, 485-490.)

WHITE, WILLIAM F. *Alice in the Wonderland of Mathematics.* The Open Court, vol. 21, no. 1, January, 1907, pp. 11-21. The Open Court Publishing Co., Chicago.

II. BOOKS FOR TEACHERS

ADAMS, JOHN (EDITOR). *The New Teaching.* Hodder and Stoughton, London, New York, Toronto, 1918. 428 pp.

The chapter on the teaching of mathematics, pp. 195-229, is by James Strachan, M.A., B.Sc., sometime Assistant Master at R. N. College, Osborne, and afterwards Chief Mathematical Master at Merchant Taylors' School. Full of suggestions.

ARCHIBALD, R. C. *Mathematics and Music.* The American Mathematical Monthly, Oberlin, Ohio, January, 1924.

ATKIN, EDITH IRENE. *The Recitation in Mathematics.* Mathematics Teacher, vol. 17, no. 8, December, 1924, pp. 459-470.

Applications that make an appeal to the pupil. An article that teachers might well read again and again. Very helpful.

BARNETT, P. A. (EDITOR). *Teaching and Organization, with Special Reference to Secondary Schools: A Manual of Practice.* Longmans, Green, and Co., London and New York, 1897.

Contains an interesting chapter on Mathematics (pp. 78-97) by R. Wormell, D.Sc., Head Master of the City Foundation Schools, London, and member of the late Royal Commission on Secondary Education.

BROWN UNIVERSITY. *Suggestions for Students of Mathematics: Mathematics and Life Activities.* Brown University, Providence, Rhode Island, Bulletin of the Department of Mathematics, Number One.

An eight page folder containing among other things a literature list and three interesting tables: (1) Occupations for which concentration in Mathematics is desirable, (2) Occupations for which concentration in Mathematics combined with other subjects is desirable, (3) Fields of work in which Mathematical Training or some knowledge of Mathematics is desirable. "After a careful statistical investigation of the best preliminary training for a student of law, Dean Roscoe Pound of the Harvard Law School advises men to take mathematics throughout the four years of their college course."

BROWN UNIVERSITY. *Facilities and Opportunities for Students Specializing in Mathematics.* Bulletin of the Department of Mathematics, Number Two.

CAMPBELL, GEORGE A. *Mathematics in Industrial Research.* The Bell System Technical Journal, American Telephone and Telegraph Co., New York, October, 1924, pages 550-557.

Contents: (1) "Selling" Mathematics to the Industries, (2) Mathematics in Electrical Communication, (3) Industrial Mathematics as a Career, (4) Training for Industrial Mathematics.

CLARK, DR. JOHN R. *Mathematics in the Junior High School.* The Gazette Press, Yonkers, New York, 1925. 160 pp. $.90.

Section titles: I Mathematics in the Junior High School, II the Larger Objectives of Mathematics in the Junior High School, III What to Teach in Junior High School Mathematics, IV How to Teach Problem Solving, V Fixing Habits, VI Newer Types of Tests and Teaching Devices, VII Graphical Methods in Grades Seven, Eight and Nine, VIII The Elementary Notions of Statistics, IX Making Mathematics Interesting, X Measuring the Results of Teaching, XI Diagnosis and Remedial Teaching, XII Generalized Values in Mathematics.

College Entrance Examination Board. *Documents No. 107 and 108.* 431 West 117th Street, New York.

Definition of the requirements in mathematics, with notes for the guidance of teachers.

Collins, J. C. *Calculation by Geometry of Astronomical Distances.* School Science and Mathematics, Chicago, vol. 20 (May, 1920), pp. 416-418.

Cosby, Byron. *Project Method in Teaching Mathematics.* School Science and Mathematics, Vol 22 (May, 1922), pp. 451-455.

Applying mathematics to everyday needs.

Crathorne, A. R. *Algebra from the Utilitarian Standpoint.* School, Science and Mathematics, vol. 16 (May, 1916), pp. 418-431.

A forceful presentation of the practical worth of even such "airy" things as complex fractions, exponents, and imaginary numbers. Should be read by every teacher.

Durell, Fletcher. *The Locus Problem in Geometry with Some Discussion of the Utilities in Geometric Study.* School Science and Mathematics, vol. 11 (January, 1911), pp. 40-46.

A practical article on how to teach the idea of a locus and how to make it interesting.

Durell, Fletcher. *Suggestions on the Teaching of Algebra with Especial Reference to the Use of Durell and Arnold's Algebra.* Charles E. Merrill Co., New York. 22 pp. Gratis.

Durell, Fletcher. *Suggestions on the Teaching of Geometry.* Charles E. Merrill Co., New York. 27 pp. Gratis.

Glazier, Harriet E. *The Mathematics of Common Things.* School Science and Mathematics, vol. 16 (November, 1916), pp. 667-674.

"It did not need the botanist to tell the tree trunk that the circular arrangement is best for its growth, nor the flower to gather its petals about the centre. It was not the scientist who taught the bird to make her nest or rear her young, nor the mathematician with his calculus that led the bees to construct the hexagonal cell for storing their honey. All these things . . . . ."

Graves and Ziegler. *The Woodsman's Handbook.* Washington: Government Printing Office, 1912. 208 pp.

Measuring the height of trees, pp. 97-99

GUGLE, MARIE. *Recreational Values Achieved through Mathematics Clubs in Secondary Schools.* Mathematics Teacher, vol. 19, no. 4 (April, 1926), pp. 214-218.

Illustrated Mathematical Talks by Pupils of the Lincoln School of Teachers College. Lincoln School of Teachers College, Columbia University, New York. 44 pp.

A beautifully illustrated booklet containing illustrated talks on such topics as "Geometry and Nature" and "Control of the Formula."

MIALL, L. C. *Thirty Years of Teaching.* Macmillan Co., New York, 1897. 250 pp.

Contains section entitled "Elementary Geometry," pp. 110-122.

MILNER, FLORENCE. *On Teaching Geometry.* D. C. Heath and Co., New York. Heath's Mathematical Monographs, No. 5. Ten cents.

NEWHALL, CHARLES W. *Recreations in Secondary Mathematics.* School Science and Mathematics, vol. 15 (April, 1915), pp. 277-293.

Contains a valuable bibliography of books and magazine articles.

NYBERG, JOS. A. *Prose Problems of Algebra.* School Science and Mathematics, vol. 20 (December, 1920), pp. 829-835

The use of a single problem to give an idea of (1) how algebra eliminates guess work and (2) a function. Imagination: (1) An equation containing parentheses and a grocery clerk unwrapping packages; (2) Simplifying radicals: compare the radical sign to a very fastidious landlord who will not shelter a tenant who is a fraction or is divisible by a perfect square (Caution: "Consider your fastidious landlord"), etc.

NYBERG, JOS. A. *The Use of Charts for Prose Problems.* School Science and Mathematics, vol. 20 (October, 1920), pp. 619-623.

The similarity between the diagram for the flower problem and the order pad of a grocery clerk, etc.

PIERCE, MARTHA. *Mathematical Recreations.* Mathematics Teacher, vol. 19, no. 1 (January, 1916), pp. 13-24.

RABOURN, SARA B. F. *Boost Mathematics.* School Science and Mathematics, vol. 16 (October, 1916), pp. 595-602.

Advertising one's product.

SIMONS, LAO GENEVRA. *Introduction of Algebra into American Schools in the Eighteenth Century.* Washington: Government Printing Office, 1924. Fifteen cents.

SYKES AND COMSTOCK. *Training Pupils to Do Algebra.* Rand McNally and Co., New York. 21 pp.

A handbook for the Sykes-Comstock Algebra.

SYKES AND COMSTOCK. *Training Pupils to Do Algebra.* Rand McNally and Co., New York. 34 pp.

TRIPP, M. O. *Some Simple Applications of Elementary Algebra to Arithmetic.* School Science and Mathematics, vol. 15 (June, 1915), pp. 496-500.

The following books and magazines are out of print. This does not in every case mean that these books are unobtainable, for there are concerns which make a business of searching for out of print books for you. I was recently fortunate enough to obtain a copy of White's "Scrap-Book of Elementary Mathematics" through one of these concerns (American Library Service, 500 Fifth Avenue, New York).

III. BOOKS THAT ARE OUT OF PRINT

BROOKS, EDWARD. *Philosophy of Arithmetic.* The Penn Publishing Co., 925-927 Filbert Street, Philadelphia. (Took over the business of The Normal Publishing Co., which was never a regular publishing house but was organized simply for handling three books by Dr. Edward Brooks.)

BRUCE, WILLIAM HERSCHEL. *The Triangle and Its Circles.* D. C. Heath and Co., New York. Heath's Mathematical Monographs. Original price, ten cents.

CALHOUN, GRACE WARD. *In Algebra Land.* (Pamphlet.) Educational Gazette, Syracuse, New York. A letter sent to this address was returned marked "Unclaimed—Cannot be located by Directory Clerk." (Address was from School Science and Mathematics, vol. 15, pp. 283 and 292.)

FRANKLAND, W. B. *The Story of Euclid.* D. Newnes, Ltd., London.

HAMPSON. *Paradoxes of Nature and Science.* E. P. Dutton and Co., New York.

KEMPE. *How to Draw a Straight Line.* Macmillan Co., New York.

KLEIN, FELIX. *Famous Problems of Elementary Geometry.* Ginn and Co., New York.

LODGE, SIR OLIVER. *Easy Mathematics.* Macmillan Co., New York.

NATURE (London) for January 22, 1891 (vol. 43, p. 273). "The Application of Geometry to Practical Life," a 30 minutes' Probationary Lecture delivered at Gresham College, December 12, 1890, by Prof. Karl Pearson. Macmillan and Co., Ltd., London.

SMITH, D. E. *Teaching of Geometry.* Ginn and Co., New York.

WHITE, WILLIAM F. *A Scrap-Book of Elementary Mathematics.* The Open Court Pub. Co., Chicago, 1908. 248 pp.

Some of the interesting topics treated by White: (1) What Does a Billion Mean?, pp. 9-10; (2) Numbers Arising from Measurement, pp. 43-46; (3) Compound Interest: If the Indians hadn't spent the $24, pp. 47-48; (4) Napier's Rods and other Mechanical Aids to Calculation, pp. 69-72; (5) Do the Axioms Apply to Equations?, pp. 76-80; (6) Algebraic Fallacies, pp. 83-88; (7) Geometric Puzzles, pp. 109-117; (8) The Three Famous Problems of Antiquity, pp. 122-125; (9) Linkages and Straight-Line Motion, pp. 136-139; (10) A Few Surprising Facts in the History of Mathematics, p. 165; (11) Quotations on Mathematics, pp. 166-167; (12) The Nature of Mathematical Reasoning, pp. 212-217; (13) Alice in the Wonderland of Mathematics, pp. 218-233; (14) Bibliography on Mathematical Recreations, pp. 234-240.

DISCUSSION

By JOHN J. HALL
West High School, Rochester, N. Y.

In a recent issue of "The Mathematics Teacher" appeared an article on the subject of the proof or reason of the rule that the product of two negative numbers is a positive number.

In this article are certain statements that might prove to be dangerous if they are taken at face value. To be more explicit, the article states the following:

> "For those pupils or teachers who cannot see or thoroughly grasp the inevitableness of the relation $(-5) \times (-3) = +15$, it should be comforting to know that they are not expected to, and *that no one can.* With this in mind, our efforts to encourage pupils to reason should be more fruitful for not expecting them to reason in situations *where it is not possible to do so.*"

Again referring to this same rule the article states "this is one of the few places where it is not possible for the pupils to reason and so understand why $(-5) \times (-3) = +15$."

Such statements as these cannot but arouse protests from those teachers who have found devices by which their classes have seen why it is perfectly logical and obvious that the product of two negative numbers does give a positive quantity.

The purpose of this paper is to submit a method of reasoning that has proved both workable and convincing in the class room for several terms with the hope that some other teacher may find it worth trying. The writer has successfully used this device in day school classes where the pupils are thirteen to fourteen years of age, as well as in night school classes composed of adults who for various reasons are taking up a belated study of high school subjects. In both kinds of classes there are always some pupils who want to know the "why" and the "wherefor." Were it not for these individuals we would possibly be still living in the Stone Age. As a group of modern teachers, can we afford to answer their questions by stating that such and such a rule

"works and so I guess the process is correct?" These pupils may accept this form of answer through politeness and good breeding, but their opinion of the teacher or of the course of study is not improved thereby.

In regard to negative numbers, were they "invented merely to make the subtraction of *B* from *A* 'general'?" Instead, did it not become necessary to recognize that there were two kinds of numbers and our knowledge of numbers increases as we learn to of positive and negative numbers is merely a convenience and possibly other terms might just as well have been chosen. We have reason to believe that the idea back of signed numbers has always existed and that our rules for them are the result of our efforts to learn how to interpret and operate with them. Do we explain irrational numbers by stating that they were invented to make the process of evolution general? Of course not, they appear whenever we construct a square and draw its diagonals. They were not labelled and recognized as irrational quantities at once but they existed just the same. There are relations between numbers and our knowledge of numbers increases as we learn to work with them. It may be that we are able to arrive at certain conclusions by deductive reasoning, but often it is necessary to do so by inductive consideration of the facts at hand. Is not the latter form of logic well recognized?

Suppose we start with a definite understanding of positive and negative numbers and present several every-day problems. If a study of these problems force us to the conclusion that the only reasonable answer obtained by multiplying two negative numbers together is a positive number and moreover if such a conclusion is perfectly consistent with the results obtained when other kinds of numbers are multiplied together, are we not justified in our contention that our reasoning has been logical? We have here an opportunity to develop the power of thinking along new lines and we owe it to ourselves as well as to the pupils to make the most of it. There is no difficulty in introducing the idea that positive and negative quantities are opposites in direction, time, financial standing, etc. We have so many practical examples of these things, that the pupils readily translate them into their every-day life. Also when we come to the fundamental operations of signed numbers there is no difficulty in presenting addi-

tion and subtraction and there should be no mystery or "ready-to-wear" rules in teaching multiplication.

The writer lays no claim to originating the idea of presenting the following suggestive problems. Like other teachers who are interested in their work and are constantly on the lookout for better ways of teaching, he has garnered a little here and a little there and this is the result.

With the understanding that "days ahead" are to be represented by positive numbers, and "days ago" by negative numbers and also that money gained is positive and money lost is negative, then, "if a merchant is losing $5 a day for every day he keeps in business, would he be better or worse off and how much, if he had given up his business three days ago?"

When thus presented the pupils will readily say that the man would be fifteen dollars ahead if he had quit three days ago and hence $(-5) \times -3)$ must be $+ 15$.

Or, if a pan of hot water is allowed to cool and the temperature drops an average of 10 degrees every hour, was it hotter or colder 3 hours ago than now? It logically follows that since the water was 30 degrees hotter, the sign of 30 must be plus. Even the poorest pupil will agree with that.

Or again; if a certain stock or bond has dropped an average of 4 points a day for the last 3 days, was it higher or lower and how much 3 days ago? And again you will get the pupil to develop in his own mind that losing 4 points a day (represented by -4) for the past 3 days (represented by -3) that the stock was 12 points higher 3 days ago which must be represented by $+12$.

In other words:

$$(1)\quad (-5) \times (-3) = +15$$
$$(2)\quad (-10) \times (-3) = +30$$
$$(3)\quad (-4) \times (-3) = +12$$

There are scores of such problems that can be given and which will satisfy the pupil (child, or adult) that the only reasonable answer we can get by multiplying two negative quantities together must be a positive quantity.

However, too many such problems might prove to be a detriment for the pupil might get the idea that you were trying to force him into something that needed to be bolstered up by a

large number of examples. It is usually much better to give a few illustrations to clinch the principle and then by your own quiet attitude of acceptance, he will be satisfied too. This attitude on the teacher's part holds true all through teaching and is also true in life. We are instantly suspicious that there is something wrong when a salesman tries to "sell" us too earnestly.

The thing to keep constantly in the pupil's mind, is that when we speak of negative or positive numbers, we are referring to opposites, and to nothing else.

NEWS NOTES

The Extension Division of the University of Iowa and the Department of Mathematics of the University held a Conference of Teachers of Mathematics at the University of Iowa, October 8-9, 1926. The program was arranged by Professor Henry L. Rietz. Among the topics appearing on the program were:

(1) Objectives in School Albegra, by Professor W. H. Wilson, University of Iowa.

(2) Interest of Pupils in High School Mathematics and Factors in the Securing of Interest, by Alfred Davis, of St. Louis.

(3) Mathematics in the Junior High School, by E. H. Taylor, Charleston, Illinois.

(4) Outstanding Errors in Mathematics made by College Freshmen in Placement Tests, by Professor Roscoe Woods, University of Iowa.

(5) Objectives in School Geometry, by Professor Wilson.

(6) Objective Examinations, by Professor G. D. Stoddard of Iowa.

(7) Importance of High School Mathematics from the Viewpoint of the Layman, by Mr. Davis.

(8) The Use of Problems in Teaching Elementary Algebra, by Mr. Taylor.

(9) On Certain Ideals of the Mathematics Teacher, by Professor Wilson.

At the October meeting of the Association of Teachers of Mathematics in New England, Professor Ralph Beatley of Harvard University read a paper on "Logarithms and Trigonometry for Infants." George W. Evans (now en route to Texas and China) contributed a paper on "Euclid and Modern Geometry."

A unique and worthy method of using a golden anniversary gift has been introduced by Professor Julius and Rosa Sachs. The sum of $20,000 presented to Professor and Mrs. Sachs, on the occasion of their golden wedding, has been established as an Endowment Fund at Teachers College, Columbia University. The Fund is to be used for the purpose of promoting by a series of prizes, the progress of secondary education in the United States.

For the year 1926-1927 the Sachs Endowment Fund offers a prize of one thousand dollars for the best essay or treatise on "The Aims and Methods of Science Teaching in the Successive Stages of a Secondary School, and the Intellectual Equipment of the Teacher That Will Enable Him to Put These Aims Into Practice." All manuscripts must be in the hands of the Dean of Teachers College, Columbia University, on or before December 1, 1926. The rules governing the competition for the Science prize may be secured from the Secretary of Teachers College, 525 West 120th Street, New York City.

SOME COMMENTS ON THE FIRST YEAR BOOK BY THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

By J. T. JOHNSON
Chicago Normal College

In this age of rapid evolution in methods of teaching mathematics, in text book making and curriculum construction no professionally spirited teacher of mathematics can well afford to be without this comprehensive treatise.

It is at the same time a historical document, a text book for teachers, a compendium of results of research, and a collection of minimal essentials for junior and senior high schools.

I. A General Survey of the Progress of Mathematics in Our High Schools in the Last Twenty-five Years
By David Eugene Smith

In accordance with the general theme of the book, David Eugene Smith summarizes the progress of mathematics in our high schools in the last twenty-five years in the opening chapter. He traces the development of the college entrance requirements up to the present time. He discusses the rise of the junior high school, the work of the National Committee, the growth of schools of education, and the reform in text book making. In the arithmetic of the junior high school a parallel column comparison is made of the topics stressed twenty-five years ago and those of importance today. The list of thirty problems from a text book of twenty-five years ago proves very interesting reading.

The progress in algebra has been mainly in elimination of certain topics and in reforming the verbal problem. In geometry the introduction of intuitive geometry is mentioned as coming into prominence and as differentiated from demonstratve geometry. It is not defined, however.

II. On the Foundations of Mathematics
By Ellakim Hastings Moore

The second chapter is a reproduction of a presidential address by E. H. Moore delivered before the American Mathematical

Society, Dec. 9, 1902. It is as interesting as it was prophetic. The reforms and changes spoken of in the first chapter by Smith were advocated in this speech. No one can estimate how much influence this address has had in directing and shaping reform in teaching mathematics in the last quarter of a century.

Moore begins by reviewing the European methods of reducing pure mathematics to a system of postulates and undefined symbols. He gives in this connection the following far-reaching and thought provoking statements and questions, "Indeed the question arises whether the abstract mathematicians in making precise the metes and bounds of logic and the special deductive sciences are not losing sight of the evolutionary character of all life-processes, whether in the individual or in the race." "What is a finite number?" "What are the permissible logical steps of deduction?" "What is science?" "What is the scientific method?" "What are the relations between the mathematical and the natural scientific processes of thought?"

In speaking of Perry, an English author and teacher of practical mathematics, he uses this very modern notion of teaching mathematics, "that the boy shall be familiar (by way of experiment, illustration, measurement, and by every possible means) with the ideas to which he applies his logic; and moreover that he should be thoroughly *interested* in the subject studied."

Lastly, under a vision, he sets forth these ideas on the pedagogy of mathematics all of which have since been tried and adopted in our best schools; correlation of different mathematical subjects, the enrichment and vitalization of material and method in primary schools, general mathematics, the junior high school, the junior college, the laboratory method and the individual method of instruction. He even advocates strongly the very modern method of employing the brighter students to help the weaker ones. He says on this, "but experience shows that just as every teacher learns by teaching so even the brightest students will find themselves much the gainers for this co-operation with their colleagues."

How prophetic was his vision let every teacher of mathematics judge!

III. Suggestions for the Solutions of an Important Problem That Has Arisen in the Last Quarter of a Century

By Raleigh Schorling

This chapter is so full of facts and tables that to get the full significance one is urged to read it in full.

Before submitting any suggestions on the solution of the above problem, the author collects abundant evidence on the fact that the achievement in scholarship in arithmetic and algebra is very low. This evidence the results of the testing movement have supplied. The tables of results give such astounding facts as the following. Out of 3,260 pupils tested at the beginning of the 7th grade, less than ½ (47.5%) were able to write .125 as a common fraction. Only 33% could complete the ratio of 1 to 2 is equal to the ratio of 5 to —. Only 74.8% answered correctly, "Write ¾ as a decimal."

On twenty of these items which 146 experienced teachers checked as those which the beginning 7th grader should know with nearly 100% accuracy, the average % of correct response was only 58.4%. It seems scarcely credible that the 8 of the 125 items which were given 90% or better correct responses were the following very simple facts:

"One dozen equals how many things?
One pound equals how many ounces?
One hour equals how many minutes?
One minute equals how many seconds?
If you have two numbers how can you find their sum?
What is $12 \div 3$?
Make drawing to show that you know the meaning of the words 'circle' and 'square'."

Results of the Woody-McCall test given to 1626 pupils in Michigan in 1924 show that only 23% of students solved correctly $7/8 \div 6$, and only 65% wrote the correct answer to

$$\begin{array}{r} 5939 \\ \times 85 \\ \hline \end{array}$$

In 9th grade algebra results from Thordike's and Hotz's tests show that none of the very elementary facts of algebra were mastered.

The author thinks that the real passing mark of our American secondary school is somewhere between 40 and 50%.

There is a need for action here. In a program for improvement the author outlines six specific measures that can safely be taken. These are timely and valuable and apply not only to the teachers of the secondary school but elementary teachers as well.

Under the Psychology of drill 20 specifications are listed and described.

The author concludes by giving the definition and aims of general mathematics.

IV. Improvement of Tests in Mathematics
By W. D. Reeve

W. D. Reeve classifies and distinguishes the various kinds of tests such as the prognostic, diagnostic, achievement tests, standardized tests and scales, the essay type of examination, and the new type of objective examination.

We have lost much time in the past, says the author, due to a lack of knowledge which tests alone have been able to supply. The unreliability in grading work in mathematics should be replaced by developing a new marking system that will rate achievement in a true relation to ability.

Much harm has also been done from a misuse of tests such as regarding the test itself as the sole aim of instruction and rating teachers from results of certain standard tests. In the future, he says, we should not attempt to standardize tests as measuring devices before we are agreed upon what abilities we wish to measure.

Tests have gone through three stages, (1) The curiosity stage, (2) The determining levels or norms of achievement, (3) The use of tests as a means for improving instruction. We are now in the third stage. In this stage the diagnostic test plays an important role. This should be followed by remedial instruction by means of scientifically prepared practice exercises or practice tests. These should become a part of the equipment of every teaching program. In this age of curriculum construction tests also play an important part in that they enable us to evaluate methods of teaching and order of teaching.

The chapter is concluded by a series of 5 objective tests in algebra and geometry, and 4 pages of selected references on tests and the teaching of secondary mathematics.

V. The Development of Mathematics in the Junior High School

By William Betz

The author begins by giving a comprehensive and helpful definition of the junior high school and its historical background. It began in Columbus, Ohio in 1908, and has grown from 6 junior high schools at that time to 1,500 or more in 1923. It has gone through three phases; in 1890 to 1900 it was guided by university administrators, in 1900 to 1910 it was promoted by public school authorities, in 1910 to the present time it is taken up by professional students of pedagogy.

The formulation of the movement has been led by such men as Bunker, Bennett, Briggs, Davis, Koos, and Glass. The 6-3-3 plan is the most popular although 9 different types of organizations have been found.

From extensive study of representative centers is found a glaring disagreement in the content of the courses. The function of intuitive geometry is often misunderstood. (It may be added here that the author did not define or make clear what is meant by intuitive geometry, hence it is still misunderstood).

The fallacious procedure of testing results before valid objecttives have been formulated is only too common.

Attention is called to the very valuable and timely attempt at a scientific listing of objectives by Professor Schorling in 1925.

The problems and contentions, he says, are mentioned but no conclusions can be drawn, no syllabus of year by year can be given because we do not yet know what should be taught in each year. A hopeful beginning has been made and the unsolved problems are confidently left to a capable future.

VI. Some Recent Investigations in Arithmetic

By Frank Clapp

Professor Clapp has here given a distinct contribution to elementary mathematics.

Three or four studies are now published on the relative difficulty of the number combinations. The method employed by Clapp was different from the others. The student was tested in two ways. First the combinations were read to the pupil in single form at the rate of one each two seconds. Pupils wrote

answers on specially prepared sheets. The combinations reduced to the automatic level were thus ascertained, and this indicated their relative learning difficulty. Secondly the pupil worked examples containing the combinations, but in this part of the test the combinations $5 + 8$, $15 + 8$, and $25 + 8$ are regarded by the author as the same. This is not borne out by experimental evidence. Hence this part of the test did not give the same relative order of difficulty as the first more reliable method. This study is also more valuable than the others in that it gives the result of all of the 100 combinations in addition, subtraction, mult plication and 90 in short division. Experimental evidence is reported to show that the reverse on an addition or multiplication combination is not the same fact as the direct form, i.e. $8 + 9$ is a different fact from $9 + 8$.

He next reports on an analysis of three text books on the relative occurrence of the easy and difficulty combinations and finds that the correlations between their frequency of occurrence and their difficulty is as follows: in addition —.452, subtraction —.329, multiplication —.384, division —.421; a rather strong negative coefficient in every case.

VII. Mathematics and the Public

By H. E. Slaught

The short section by Professor Slaught comes as a stimulating and satisfying consolation near the end of book. We are made to see that mathematics is not dead. The last quarter of a century has seen the creation of the Mathematical Association of America whose organ is the American Mathematical Monthly, and the organization of the National committee on Mathematical Requirements. In the University field has developed an American School of Mathematical Research. This is largely fostered by the American Mathematical Society. There are four research Journals, the Bulletin and the Transactions of the American Mathematical Society, The American Journal of Mathematics and the Annals of Mathematics.

The National Resarch Council and the National Academy of Sciences are now devoting part of their time to research in mathematics. The statistical method in education is bringing into light the new need for mathematics.

Not only the mathematics specialists, but the general public as well is becoming aware of the importance of mathematics. Not only did the war serve to elevate the importance of mathematics, but in times of peace, industry and health promoting institutions have seen fit to include in their programs research in mathematics. The Rockefeller Foundation has recently extended its fellowships to include mathematical research.

Dr. Slaught, himself, is largely responsible for creating and keeping alive this national interest in mathematical research by being an active member in many nation-wide organizations.

VIII. Recreational Values Achieved Through Mathematics Clubs in Secondary Schools

By Marie Gugle and Others

In this section the authors have brought to our attention the richness of mathematics as a field for recreation and puzzles. This is in accordance with one of the seven objectives—the proper use of leisure time—as given in the Cardinal Principles of Secondary Education.

The first mathematical puzzle is attributed to Ahmes in 2000 B. C. Magic squares were known to Arabs and Hindus. The three best sources for this material are Ball's Mathematical Recreations, White's Scrap Book and Jones' Mathematical Wrinkles.

Before 1900 there were no mathematics clubs in high schools of the U. S. By 1916 they were prevalent throughout the entire country.

The purposes of and the qualifications for membership in mathematics clubs are clearly set forth. Directions for organizing and a complete series of programs for 7th, 8th and 9th grades are given. Every valuable nucleus list of books for a club library is given and a discussion of mathematics plays written by students. The section as a whole gives just the information needed for anyone wishing to start a mathematics club in any grade from 7 to 12.

IX. Mathematics Books Published in Recent Years For Our Schools and For Our Teachers

By Edwin W. Schreiber

As a concluding chapter this gives further information in a 9-page list of references of mathematics books published since 1920.

The following 6 facts concerning each reference are given:

What is the exact title of the book?
Who is the author and where does he teach?
How many pages in the book and how large is it?
When was it published?
What is the price?
Who publishes the book?

The references are distributed as follows:

11	Junior High School texts, all but 2 in 3-book series.
13	Regular elementary texts, 3-book series.
6	General Mathematics.
5	Practical Mathematics.
23	Algebras I.
9	Algebras II.
5	Algebras I and II.
13	Plane Geometries.
5	Solid Geometries.
5	Plane and Solid Geometries.
8	Trigonometries.
3	College Algebras.
1	Calculus.
10	Histories of Mathematics.
24	Teaching of Mathematics.
141	All published since 1920.

NEW BOOKS

The Law of Diminishing Returns—By W. J. Spillman and Emil Lang; World Book Company. Pp. 178, 1925.

Statistical Analysis—By Edmund E. Day; the Macmillan Company. Pp. 459, 1925.

Modern Methods in High School Teaching—By Harl R. Douglass; Houghton Mifflin Company. Pp. 544, 1926.

Practical Psychology—By Edward Stevens Robinson; the Macmillan Company. Pp. 479, 1926.

Arithmetics—Books I, II, III. By Samuel Edwin Weber, Charles Dison Kooch, and Katherine Ellen Moran; Christopher Sower Company. Pp. 254, 1925.

Education of Gifted Children—By Lulu M. Stedman; World Book Company. Pp. 192, 1924.

Algebra—By William Raymond Longley and Harry Brooks Marsh; the Macmillan Company. Pp. 577, 1926.

Advanced Algebra—By Edward I. Edgerton and Perry A. Carpenter; Allyn and Bacon. Pp. 377, 1925. Price $1.40.

Plane Trigonometry—By Lloyd L. Smail; McGraw-Hill Book Company. Pp. 192, 1926.

Junior High School Mathematics—Book III, Revised. By George Wentworth, David Eugene Smith and Joseph Clifton Brown; Ginn and Company. Pp. 338, 1926. Price $1.20.

Exercises and Tests in Algebra—By David Eugene Smith, William David Reeve and Edward Longworth Morss; Ginn and Company, 1926.

A Second Book in Algebra—By Howard B. Baker; D. Appleton and Company. Pp. 265, 1926.

Principles of Bookkeeping and Business—By Charles E. Bowman and Atlee L. Perry. Edited by Frederick G. Nichols. American Book Company. Pp. 287, 1926.

An elementary high school course which teaches pupils how to interpret business records as well as how to make such records. Both record keeping and record analysis are emphasized. The subject is treated through a modified balance sheet approach. The fundamental principles of accounts are presented in a manner that will aid the pupil in acquiring a thorough understanding of those principles. Suitable problem material is provided for application and drill.

A Geometry Reader—By Julius J. H. Hayn, Head of Mathematics Department, Masten Park High School, Buffalo, New York. Published by Bruce Publishing Company, 1925.

The object of the author in writing this book is to produce a readable text for the teaching of geometry. In his own words: "Pussy will not eat mush, but is very fond of a freshly opened can of salmon. Mix the two, and pussy eats both with evident relish."

The Introduction and Book I differ from the usual text in their content and method of presentation. The first part of the introduction deals with a story of Socrates and the Boy and passes very soon to the algebraic applications of the formulas used in the entire course in geometry. After an exhaustive study of these, there is a discussion of the instruments of geometry, together with simple construction exercises, the theorems of limits and a few definitions.

Since this a reader, rather than the usual geometry text, Book I begins with the explanation of angles, lines and other terms and gives as the first eight principles what are usually considered as postulates or as corollaries to postulates. Proposition Nine, which is the first real proposition is: "If two parallel lines are cut by a third, the exterior interior angles are equal." Following this we have the axioms, more definitions, and the remainder of the theorems. It is interesting to note that the propositions are written in paragraph form with the reasons in the last part of the sentence after "because." The author ob-

jects to parallel columns of "steps" and "reasons" for "if done too frequently, it becomes about as monotonous as anything can be and robs a splendid demonstration of all its beauty." The congruency of triangle propositions start with Proposition 25, as: "Two right triangles are equal, if the sides of the right angle are equal each to each," and is proved by the superposition method. Two ways of proving general triangls are given: one, by dividing it into right triangles and using the right triangle propositions for proof; and the other, the usual method of superposition. The parallelogram propositions, constructions, unequal lines and angles, and locus propositions follow. As a chapter in a reader ends with a summary, so the conclusion of Book I sums up the contents of that book and introduces the following books in a general way.

The subject matter of the rest of the books is much the same as that of other books: Book II deals with circles; Book III with proportion; Book IV with areas and Book V with regular polygons.

There are, in all, one hundred and eighty-eight propositions, sixty-seven of which are in Book I. Of these one hundred and eighty-eight, proofs are given for about half of them with hints to the proof of the others. Scattered throughout the book are numerous rather easy exercises.

Hence we see that this text differs from others in its algebraic introduction and in the changes made in Book I. There is an attempt to make the subject interesting by occasional short stories of mathematicians, by a topical analysis at the beginning of each Book, and by an explanation of the transition from one topic to another.

Ruth W. Campbell
Buffalo, New York.

A Laboratory Plane Geometry—By William A. Austin, Head of the Department of Mathematics, Venice Polytechnic High School, Los Angeles, California. Published by Scott Foresman and Company, 1926.

The author of this book appreciates the value of the law of learning by doing. By this laboratory method he seeks to "work geometry into the understanding of the pupil." Instead of be-

ing given certain facts and told to prove them, the author would have his pupils make constructions according to specific directions, and then by measurements and computations discover seeming truths for themselves. The pupils would next seek proofs for these seeming truths, and if proven, state them as general truths in the form of propositions. According to the plan of the book the pupils are next given many and varied applications to fix the proposition in mind, practical problems for this purpose abounding throughout the book. Numerous thought provoking questions and numerical exercises are introduced.

Instead of the dry formal definitions to which the pupil is usually introduced when he first begins geometry, the author seeks to arouse the interest of the pupil by giving him "real work at real things."

He leaves him to learn the meaning of the terms as they come incidentally while he works with the constructions.

By the method used in this book individual differences are recognized and taken care of principally by means of supervised study.

The material used in this book has been worked out in the classroom by the author and others through a period of over five years.

The "Fundamental Theorems and Constructions" of the 1923 report of the National Committee of Mathematical Requirements are indicated by bold-face italics, while those for which the College Entrance Examination Board will hold the pupils primarily responsible are indicated by a star.

The book is exceptionally well bound and attractive in appearance.

The reviewer believes that the methods used in this book will not only arouse the interest of the pupil and cause him to see the practical side of the subject, but will also enable him to discover geometrical facts and proofs for himself.

BESS L. SCOTT
Dallas, Texas.

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(Continued from October issue)

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Roosevelt High School, 23rd and Cranberry Streets, Erie, Pennsylvania.
Ruth Flory, 81 N. Courtlandt St., E. Stroudsburg, Pennsylvania.
Thomas Moore, 1128 Fillmore St., Frankford, Philadelphia, Pennsylvania.
Department of Mathematics, Frankford High School, Frankford, Philadelphia, Pennsylvania.
Emma Roth, 5128 Saul St., Frankford, Philadelphia, Pennsylvania.
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Dr. A. H. Wilson, Haverford College, Haverford, Pennsylvania.
Anna D. Klauss, 41 Trinidad Ave., Hershey, Pennsylvania.
A. Mathewson, Box 161, Hershey, Pennsylvania.
Mr. Port Ecles, Junior High School Bldg., Homestead, Pennsylvania.
State Normal School, Indiana, Pennsylvania.
Elvira Thomas, Cochran Junior High School, Johnstown, Pennsylvania.
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Irvin O. Noll, 59 West Essex Ave., Lansdowne, Pennsylvania.
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(To be continued)